A Symbolic Algorithm for the Computation of Periodic Orbits in Non–Linear Differential Systems

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Abstract The Poincaré–Lindstedt method in perturbation theory is used to compute periodic solutions in perturbed differential equations through a nearby periodic orbit of the unperturbed problem. The adaptation of this technique to systems of differential equations of first order could produce meaningful advances in the qualitative analysis of many dynamical systems. In this paper, we present a new symbolic algorithm as well as a new symbolic computation tool to calculate periodic solutions in systems of differential equations of first order. The algorithm is based on an optimized adaptation of the Poincaré–Lindstedt technique to differential systems. This algorithm is applied to compute a periodic solution in a Lotka–Volterra system.

Keywords: periodic orbits, Poincaré–Lindstedt, Lotka–Volterra, Poisson series, symbolic computation.

1 Introduction

Periodic orbits play a key role in understanding dynamical systems. Their rigorous computation and the precise knowledge of their properties has made them very important for studying the behavior of many dynamical systems of physical interest. In "Les méthodes nouvelles de la Mécanique Céleste" [17], Poincaré considers the determination of periodic solutions by series expansion with respect to a small parameter. Consider for instance the equation $\ddot{x} + x = \epsilon f(x)$, and suppose that an isolated periodic solution exists for $0 < \epsilon \ll 1$, and if $\epsilon = 0$ all solutions are periodic. Under certain conditions Poincaré proves that we can describe the periodic solution by a convergent series in entire powers of ϵ, where the coefficients are bounded functions of time. The Poincaré–Lindstedt expansion is a classical perturbation method used to continue a periodic orbit with respect to a small perturbation parameter, when fixing the amplitude (or the energy) of the system. This method has been used extensively to the analysis of a wide variety of systems in many branches of science: from galactic ([21], [22]) to atomic models [16], and covering also applications in population biology, ecology and mathematical biology [5]. Nowadays, many researchers ([2],[3], [4], [8], [19] and [24] to cite some examples) make use of this method to study dynamical systems. Buonomo [2] uses the Poincaré expansion theorem in order to compute the periodic solution of the van der Pol equation, Hu & Xiong [8] explore the capabilities of the Poincaré–Lindstedt method to study the cubic Duffing equation, and the quintic Duffing equation is also examined with this technique by Ramos [19]. As stated in [4] and [24], the only algorithms capable of computing high–precision periodic orbits are based on the Poincaré–Lindstedt series method. In [24], the author presents the most accurate computation of Hill's orbit of lunation since its justly celebrated discovery in 1878 with the help of the Poincaré–Lindstedt technique. The Poincaré–Lindstedt technique has traditionally been used to deal with oscillators with power–law potentials. In [1], Bhattacharjee et al. show how this method can be extended to deal with molecular potentials for which the frequency goes to zero as the energy approaches zero. The extension requires the use of an asymptotic analysis which is combined with perturbation theory. They also point out that the time period obtained in their study can be used to obtain the quantum mechanical energy levels of these potentials within the Bohr–Sommerfeld scheme.

As described before, the Poincaré–Lindstedt technique is only to solve nonlinear differential equations depending on a small parameter. To overcome this limitation, some modified Poincaré–Lindstedt–type methods have been proposed in recent years. Chen & Cheung [3] employ a modification of the Poincaré–Lindstedt technique to analyze some strongly non–linear oscillators of two degree of freedom. The proposed method is based on a parameter transformation such that a strongly nonlinear system with a large

parameter is transformed into a small parameter system. In [6], He introduces a new Poincaré–Lindstedt method based on the expansion of a constant, rather than the nonlinear frequency, in powers of the expanding parameter to avoid the ocurrence of secular terms in the perturbation series solution. A detailed review of some other Poincaré–Lindstedt–type methods can be found in [19].

The application of the Poincaré–Lindstedt technique requires massive symbolic computation. The most common perturbation methods tend to produce expressions containing thousands of terms, and their treatment with general symbolic packages becomes a time–consuming task [7]. Specific symbolic computation packages avoid this inconvenience working with simple data structures and algorithms. In [11], Navarro describes a symbolic processor which could be a very useful tool to implement the Poincaré–Lindstedt method. This package works with the so called modified quasipolynomial. In that paper, the authors described a general algorithm to construct the general solution to a second order linear differential equation of constant coefficients. This package has been used to implement the Poincaré–Lindstedt technique as detailed in [12], as well as the asymptotic expansion method [13].

However, the efforts devoted to the application of the Poincaré–Lindstedt technique to systems of differential equations presenting a periodic orbit have not been so abundant. In [23], an approximation to the periodic solutions of the general Lotka–Volterra prey–predator system is obtained using the Poincaré–Lindstedt method. The computation of periodic solutions in Lotka–Volterra systems is an open problem where the Poincaré–Lindstedt method could play a key role in the computation of periodic orbits and the understanding of the way the phase space is structured not only in two species systems. In [9], Li obtains sufficient conditions for the existence of at least four positive almost periodic solutions to two species parasitical system with impulsive effects and harvesting terms, by applying MawhinÕs continuation theorem of coincidence degree theory. Camelia Pop et al. [18] study the existence of periodic solutions in the Lotka–Volterra system from the mechanical geometry point of view. Miao and Ke [10] investigate a generalized Gilpin–Ayala competition system which is more general and more realistic than the classical Lotka–Volterra competition system. By the fixed–point theorem and differential mean value, some sufficient conditions guaranteeing the existence, uniqueness and exponential stability of positive periodic solutions for a generalized Gilpin–Ayala competition system with time delays are given. In [14], Pang, Xu and Zhang give some sufficient conditions for the existence of at least two positive almost periodic solutions of harvesting predator-prey model with Holling III type functional response and time delays by means of the same procedure followed by Miao and Ke [10]. Pei and Wang [15] study dynamics of a logistical predator–prey system with state feedback control and a general functional responses. By using the Poincare map, some conditions for the existence and stability of semi-trivial solution and positive periodic solution are obtained. They show that a chaotic solution is generated via a cascade of period doubling bifurcations.

The aim of this paper is to present a general algorithm for implementing the standard Poincaré–Lindstedt method to systems of differential equations of first order. We also present here the specific symbolic tool needed to implement this technique, as well as an application of this algorithm to a simple Lotka–Volterra system to clarify some aspects related to the application of the method.

2   Adaptation of the Poincaré–Lindstedt Method for Systems

Let us consider the problem defined by the following nonlinear differential system of first order,

$$
\begin{aligned}
\dot{x} + y &= \epsilon\, f(x, y)\,, \\
\dot{y} - x &= \epsilon\, g(x, y)\,,
\end{aligned}
\tag{1}
$$

where $0 < \epsilon \ll 1$ is a small parameter and functions $f(x, y)$ and $g(x, y)$ can be arranged as follows,

$$
\begin{aligned}
f(x, y) &= \sum_{0 \le q \le M} \sum_{0 \le \nu \le q} f_{\nu, q-\nu}\, x^{\nu} y^{q-\nu}\,, \\
g(x, y) &= \sum_{0 \le q \le M} \sum_{0 \le \nu \le q} g_{\nu, q-\nu}\, x^{\nu} y^{q-\nu}\,,
\end{aligned}
\tag{2}
$$

where $f_{\nu, q-\nu}, g_{\nu, q-\nu} \in \mathbb{R}$ for $0 \le q \le M$, $0 \le \nu \le q$ and $M \in \mathbb{N}$.

If the unperturbed system ($\epsilon = 0$) has periodic solutions and ϵ is a measure of the size of the perturbing terms, then the trajectories for the full system will remain pretty close to those of the non–perturbed system, for any finite period of time $t_0 < t < t_0 + \alpha$ ($\alpha > 0$) with an error not larger than $O(\alpha)$. In general, even a small perturbation is enough to destroy periodicity, that is, nonlinearity will finish with most of the periodic orbits of the unperturbed system, but some of them may persist. The Poincaré–Lindstedt technique is used to find those periodic solutions by expanding the solution of the system in the form

$$
\begin{aligned}
x(t) &= x_0(T) + \epsilon\, x_1(T) + \epsilon^2\, x_2(T) + \cdots, \\
y(t) &= y_0(T) + \epsilon\, y_1(T) + \epsilon^2\, y_2(T) + \cdots,
\end{aligned}
\tag{3}
$$

where $x_\nu = x_\nu(T)$ and $y_\nu = y_\nu(T)$ are 2π–periodic in T, and $T = \omega t$ is the stretched time variable, with

$$
\omega = 1 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \cdots,
\tag{4}
$$

being ω_ν real constants. Thus, the nonlinear period is $2\pi/\omega$.

To apply this technique, one has to start by rewriting (1) in terms of the new independent variable T, to obtain

$$
\begin{aligned}
\omega x' + y &= \epsilon \sum_{0 \le q \le M} \sum_{0 \le \nu \le q} f_{\nu, q-\nu}\, x^\nu y^{q-\nu}, \\
\omega y' - x &= \epsilon \sum_{0 \le q \le M} \sum_{0 \le \nu \le q} g_{\nu, q-\nu}\, x^\nu y^{q-\nu}.
\end{aligned}
\tag{5}
$$

Here, $\dot{\,}$ stands for d/dt and $'$ for d/dT. If expansions (3) and (4) are substituted into (5), and terms in equal powers of ϵ are collected, we get an equation for each order of the approximation in the expansions (3). In order to simplify the expression of these equations, let us introduce here the following notation: S_ν denotes the ν–th order coefficient of the expansion of S, so that

$$
S = S_0 + \epsilon S_1 + \epsilon^2 S_2 + \cdots.
$$

For instance, if $S = x^2$, then $(x^2)_0 = x_0 x_0$, $(x^2)_1 = 2 x_0 x_1$, and in general, $(x^2)_q = \sum_{0 \le \nu \le q} x_\nu x_{q-\nu}$. This notation eases the way to express the formulae for the computation of the coefficients of the expansion of the solution at any order.

The solution to (1) is constructed from the order zero, which corresponds with the unperturbed problem, and can be written as

$$
\begin{aligned}
x_0' + y_0 &= 0, \\
y_0' - x_0 &= 0.
\end{aligned}
\tag{6}
$$

The first order system is given by

$$
\begin{aligned}
x_1' + y_1 &= \sum_{0 \le q \le M} \sum_{0 \le \nu \le q} f_{\nu, q-\nu} x_0^\nu y_0^{q-\nu} - \omega_1 x_0', \\
y_1' - x_1 &= \sum_{0 \le q \le M} \sum_{0 \le \nu \le q} g_{\nu, q-\nu} x_0^\nu y_0^{q-\nu} - \omega_1 y_0'.
\end{aligned}
\tag{7}
$$

The order Q of the expansion is obtained by solving the system

$$
\begin{aligned}
x_Q' + y_Q &= \sum_{0 \le q \le M} \sum_{0 \le \nu \le q} f_{\nu, q-\nu} \left[x^\nu y^{q-\nu} \right]_{Q-1} - \sum_{1 \le \nu \le Q-1} x_\nu' \omega_{Q-\nu} - \omega_Q x_0', \\
y_Q' - x_Q &= \sum_{0 \le q \le M} \sum_{0 \le \nu \le q} g_{\nu, q-\nu} \left[x^\nu y^{q-\nu} \right]_{Q-1} - \sum_{1 \le \nu \le Q-1} y_\nu' \omega_{Q-\nu} - \omega_Q y_0'.
\end{aligned}
\tag{8}
$$

At each order p of the perturbation method, one has to calculate x_p, y_p and ω_p from the equation above, but also x_p', y_p' and the collection of products $(x^{\nu_1} y^{\nu_2})_p$ for each $\nu_1, \nu_2 \in \mathbb{Z}$ such that $0 \le \nu_1, \nu_2 \le M$, in order to compute the right–hand side of equation (8) for the $(p+1)$–th order of the perturbation method. At the p–th order of the Poincaré–Lindstedt approximation, one first fits the value of ω_p to assure that no secular terms exist, expressing it as a function of some constants which depend on the initial conditions of the problem. Once ω_p has been obtained, x_p and y_p can be computed by solving the system (8). In the next section, we describe a procedure for the implementation of the Poincaré–Lindstedt method.

3   Application to Lotka–Volterra Systems

In this section we apply the algorithm described above to a Lotka–Volterra system in order to understand the mathematical object we deal with when we apply the Poincaré–Lindstedt method. The Lotka–Volterra equations constitute a differential system for modelling the interaction between two species. These equations have one non–trivial equilibrium solution and periodic solutions forming close orbits around this point in the phase plane. The Lotka–Volterra system can be written as

$$\frac{dX}{d\tau} = X(\tau)(a - bY(\tau)),$$
$$\frac{dY}{d\tau} = Y(\tau)(cX(\tau) - d), \tag{9}$$

where $a, b, c, d > 0$. Now, defining the variables

$$\xi(t) = \frac{c}{d}X(\tau), \quad \eta(t) = \frac{b}{a}Y(\tau), \quad t = a\tau, \quad \alpha = \frac{d}{a},$$

we get

$$\frac{d\xi(t)}{dt} = \xi(t)(1 - \eta(t)),$$
$$\frac{d\eta(t)}{dt} = -\alpha\eta(t)(1 - \xi(t)). \tag{10}$$

This system presents an equilibrium solution at $\xi(t) = \eta(t) = 1$. Now, if we perturb the system around this point,

$$\xi(t) = 1 + \epsilon x(t), \quad \eta(t) = 1 + \epsilon y(t),$$

with $\epsilon \ll 1$, and taking $\alpha = 1$, we get

$$\dot{x}(t) = -y(t) - \epsilon x(t)y(t),$$
$$\dot{y}(t) = x(t) + \epsilon x(t)y(t). \tag{11}$$

Now, we will apply the algorithm introduced in the last section to this system.

3.1   Order Zero

The solution to (1) is constructed from the order zero, which corresponds with the unperturbed problem, and can be written as

$$x_0' + y_0 = 0,$$
$$y_0' - x_0 = 0. \tag{12}$$

The solution to (12) is given by

$$x_0(t) = A_0 \cos T - B_0 \sin T,$$
$$y_0(t) = A_0 \sin T + B_0 \cos T, \tag{13}$$

where A_0 and B_0 are constants depending on the initial conditions of the problem. The derivatives of x_0 and y_0 with respect to T are

$$x_0'(t) = -A_0 \sin T - B_0 \cos T,$$
$$y_0'(t) = -B_0 \sin T + A_0 \cos T. \tag{14}$$

3.2   First Order

The first order system is given by

$$
\begin{aligned}
x_1' + y_1 &= -x_0 y_0 - \omega_1 x_0' \,, \\
y_1' - x_1 &= x_0 y_0 - \omega_1 y_0' \,.
\end{aligned}
\tag{15}
$$

Substituting (13) and (14) into (21), we get the system

$$
\begin{aligned}
x_1' + y_1 &= -\frac{1}{2}(A_0^2 - B_0^2)\sin 2T - A_0 B_0 \cos 2T + \omega_1 A_0 \sin T + \omega_1 B_0 \cos T \,, \\
y_1' - x_1 &= \frac{1}{2}(A_0^2 - B_0^2)\sin 2T + A_0 B_0 \cos 2T + \omega_1 B_0 \sin T - \omega_1 A_0 \cos T \,.
\end{aligned}
\tag{16}
$$

Now, we set the value of ω_1 to assure that resonant terms dissapear. To that purpose, we apply the following result: the system

$$
\begin{aligned}
x' + y &= A \sin T + B \cos T + \sum_{m \geq 2} A_m \sin(mT) + B_m \cos(mT) \,, \\
y' - x &= C \sin T + D \cos T + \sum_{m \geq 2} C_m \sin(mT) + D_m \cos(mT) \,,
\end{aligned}
$$

has periodic solutions if, and only if, $A - D = 0$ and $B + C = 0$. Thus, we get that $2\omega_1 A_0 = 2\omega_1 B_0 = 0$, so $\omega_1 = 0$. Equation (22) now reads

$$
\begin{aligned}
x_1' + y_1 &= -\frac{1}{2}(A_0^2 - B_0^2)\sin 2T - A_0 B_0 \cos 2T \,, \\
y_1' - x_1 &= \frac{1}{2}(A_0^2 - B_0^2)\sin 2T + A_0 B_0 \cos 2T \,.
\end{aligned}
\tag{17}
$$

Now, we have to compute the particular solution of (23), arising from the higher harmonics on the right–hand side of the system. No complementary solution is calculated, since the general solution of the homogeneous system includes two arbitrary contants, and we will take this into account when determing the value of A_0 and B_0 from the initial conditions. The particular solution of (23) is

$$
\begin{aligned}
x_1 &= \frac{1}{6}\left(A_0^2 - B_0^2 - 4A_0 B_0\right)\sin 2T + \frac{1}{3}\left(A_0^2 - B_0^2 + A_0 B_0\right)\cos 2T \,, \\
y_1 &= \frac{1}{6}\left(A_0^2 - B_0^2 + 4A_0 B_0\right)\sin 2T - \frac{1}{3}\left(A_0^2 - B_0^2 - A_0 B_0\right)\cos 2T \,.
\end{aligned}
\tag{18}
$$

Taking derivatives with respect to T, we get

$$
\begin{aligned}
x_1' &= -\frac{2}{3}\left(A_0^2 - B_0^2 + A_0 B_0\right)\sin 2T + \frac{1}{3}\left(A_0^2 - B_0^2 - 4A_0 B_0\right)\cos 2T \,, \\
y_1' &= \frac{2}{3}\left(A_0^2 - B_0^2 - A_0 B_0\right)\sin 2T + \frac{1}{3}\left(A_0^2 - B_0^2 + 4A_0 B_0\right)\cos 2T \,.
\end{aligned}
\tag{19}
$$

Thus, the solution to the first order is given by

$$
\begin{aligned}
x(t) &= x_0(T) + \epsilon\, x_1(T) = \\
&= A_0 \cos T - B_0 \sin T + \epsilon\left(\frac{1}{6}\left(A_0^2 - B_0^2 - 4A_0 B_0\right)\sin 2T + \frac{1}{3}\left(A_0^2 - B_0^2 + A_0 B_0\right)\cos 2T\right) \,, \\
y(t) &= y_0(T) + \epsilon\, y_1(T) = \\
&= A_0 \sin T + B_0 \cos T + \epsilon\left(\frac{1}{6}\left(A_0^2 - B_0^2 + 4A_0 B_0\right)\sin 2T - \frac{1}{3}\left(A_0^2 - B_0^2 - A_0 B_0\right)\cos 2T\right) \,.
\end{aligned}
\tag{20}
$$

Here, $T = \omega t$, and

$$
\omega = 1 + \epsilon\omega_1 = 1 \,.
$$

3.3   Second Order

The second order system is given by

$$x_2' + y_2 = -x_0 y_1 - x_1 y_0 - \omega_1 x_1' - \omega_2 x_0' \,,$$
$$y_2' - x_2 = x_0 y_1 + x_1 y_0 - \omega_1 y_1' - \omega_2 y_0' \,. \tag{21}$$

Substituting (18) and (19) into (21), we get

$$
\begin{aligned}
x_2' + y_2 &= \\
&= -\frac{1}{12}\left(A_0^3 - A_0^2 B_0 + A_0 B_0^2 - B_0^3\right)\sin T - \frac{1}{12}\left(A_0^3 + A_0^2 B_0 + A_0 B_0^2 + B_0^3\right)\cos T + \\
&\quad + \frac{1}{4}\left(A_0^3 + 3A_0^2 B_0 - 3A_0 B_0^2 - B_0^3\right)\sin 3T - \frac{1}{4}\left(A_0^3 - 3A_0^2 B_0 - 3A_0 B_0^2 + B_0^3\right)\cos 3T - \\
&\quad - \omega_2\left(-A_0\sin T - B_0\cos T\right)\,, \\
y_2' - x_2 &= \\
&= \frac{1}{12}\left(A_0^3 - A_0^2 B_0 + A_0 B_0^2 - B_0^3\right)\sin T + \frac{1}{12}\left(A_0^3 + A_0^2 B_0 + A_0 B_0^2 + B_0^3\right)\cos T - \\
&\quad - \frac{1}{4}\left(A_0^3 + 3A_0^2 B_0 - 3A_0 B_0^2 - B_0^3\right)\sin 3T + \frac{1}{4}\left(A_0^3 - 3A_0^2 B_0 - 3A_0 B_0^2 + B_0^3\right)\cos 3T - \\
&\quad - \omega_2\left(-B_0\sin T + A_0\cos T\right)\,.
\end{aligned}
\tag{22}
$$

Now we compute the value of ω_2 to assure that resonant terms dissapear. Then, we obtain the following equations,

$$-\frac{1}{6}A_0^3 - \frac{1}{6}A_0 B_0^2 + 2\omega_2 A_0 = 0\,,$$
$$-\frac{1}{6}A_0^2 B_0 - \frac{1}{6}B_0^3 + 2\omega_2 B_0 = 0\,.$$

Thus, the value of ω_2 is given by

$$\omega_2 = \frac{1}{12}\left(A_0^2 + B_0^2\right)$$

to have 2π periodic solutions in T. Now, equation (22) is written as

$$
\begin{aligned}
x_2' + y_2 &= \\
&= \frac{1}{12}\left(A_0^2 B_0 + B_0^3\right)\sin T - \frac{1}{12}\left(A_0^3 + A_0 B_0^2\right)\cos T + \\
&\quad + \frac{1}{4}\left(A_0^3 + 3A_0^2 B_0 - 3A_0 B_0^2 - B_0^3\right)\sin 3T - \frac{1}{4}\left(A_0^3 - 3A_0^2 B_0 - 3A_0 B_0^2 + B_0^3\right)\cos 3T\,, \\
y_2' - x_2 &= \\
&= \frac{1}{12}\left(A_0^3 + A_0 B_0^2\right)\sin T + \frac{1}{12}\left(A_0^2 B_0 + B_0^3\right)\cos T - \\
&\quad - \frac{1}{4}\left(A_0^3 + 3A_0^2 B_0 - 3A_0 B_0^2 - B_0^3\right)\sin 3T + \frac{1}{4}\left(A_0^3 - 3A_0^2 B_0 - 3A_0 B_0^2 + B_0^3\right)\cos 3T\,.
\end{aligned}
\tag{23}
$$

The particular solution of this system can be easily computed via the method of undetermined coefficients as done in the previous step. As before, a complementary solution is computed, since the general solution of the homogeneous system includes two arbitrary contants, and we will take this into account when determing the value of A_0 and B_0 from the initial conditions. The solution to (23) reads

$$
\begin{aligned}
x_2 &= -\frac{1}{12}(A_0^3 + A_0 B_0^2)\sin T - \frac{1}{12}(A_0^2 B_0 + B_0^3)\cos T - \\
&\quad - \frac{1}{16}\left(2A_0^3 - 3A_0^2 B_0 - 6A_0 B_0^2 + B_0^3\right)\sin 3T - \frac{1}{16}\left(A_0^3 + 6A_0^2 B_0 - 3A_0 B_0^2 - 2B_0^3\right)\cos 3T\,, \\
y_2 &= \frac{1}{16}\left(A_0^3 - 6A_0^2 B_0 - 3A_0 B_0^2 + 2B_0^3\right)\sin 3T + \frac{1}{16}\left(2A_0^3 + 3A_0^2 B_0 - 6A_0 B_0^2 - B_0^3\right)\cos 3T\,.
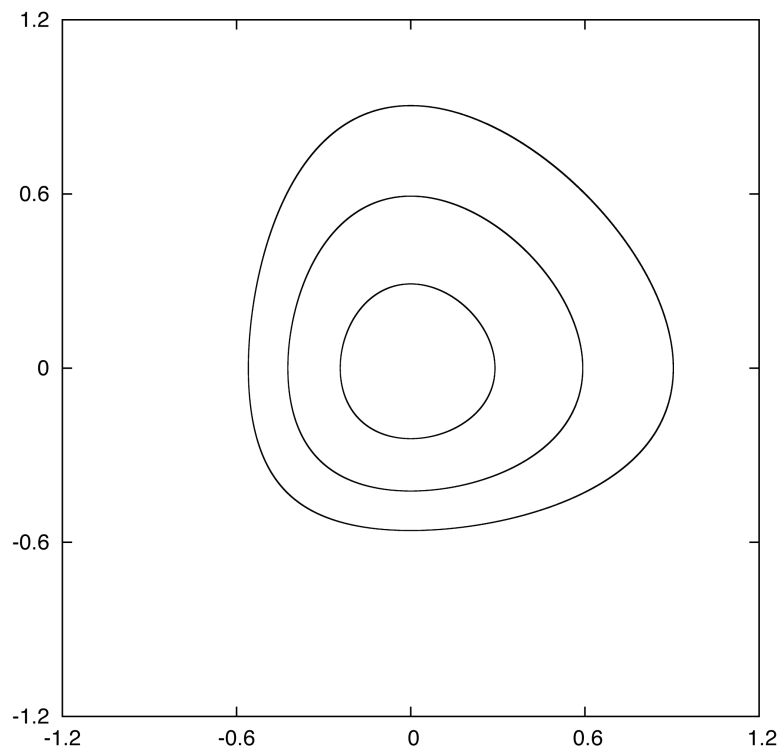\end{aligned}
\tag{24}
$$

Figure 1. Some of the periodic solutions computed through a second order approximation.

Then, the solution to the second order is

$$
\begin{aligned}
x(t) = x_0(T) + \epsilon\, x_1(T) + \epsilon^2\, x_2(T) = \\
= A_0 \cos T - B_0 \sin T + \\
+ \epsilon \left(\frac{1}{6} \left(A_0^2 - B_0^2 - 4A_0 B_0 \right) \sin 2T + \frac{1}{3} \left(A_0^2 - B_0^2 + A_0 B_0 \right) \cos 2T \right) + \\
+ \epsilon^2 \left(-\frac{1}{12}(A_0^3 + A_0 B_0^2) \sin T - \frac{1}{12}(A_0^2 B_0 + B_0^3) \cos T - \right. \\
\left. - \frac{1}{16} \left(2A_0^3 - 3A_0^2 B_0 - 6A_0 B_0^2 + B_0^3 \right) \sin 3T - \frac{1}{16} \left(A_0^3 + 6A_0^2 B_0 - 3A_0 B_0^2 - 2B_0^3 \right) \cos 3T \right),
\end{aligned}
$$

$$
\begin{aligned}
y(t) = y_0(T) + \epsilon\, y_1(T) + \epsilon^2\, y_2(T) = \\
= A_0 \sin T + B_0 \cos T + \\
+ \epsilon \left(\frac{1}{6} \left(A_0^2 - B_0^2 + 4A_0 B_0 \right) \sin 2T - \frac{1}{3} \left(A_0^2 - B_0^2 - A_0 B_0 \right) \cos 2T \right) + \\
+ \epsilon^2 \left(\frac{1}{16} \left(A_0^3 - 6A_0^2 B_0 - 3A_0 B_0^2 + 2B_0^3 \right) \sin 3T + \frac{1}{16} \left(2A_0^3 + 3A_0^2 B_0 - 6A_0 B_0^2 - B_0^3 \right) \cos 3T \right).
\end{aligned}
\tag{25}
$$

Here, $T = \omega t$, and

$$
\omega = 1 + \epsilon \omega_1 + \epsilon^2 \omega_2 = 1 + \epsilon^2 \frac{1}{12} \left(A_0^2 + B_0^2 \right).
$$

In Figure 1, we show some of the periodic solutions for several values of the initial conditions.

All these equations give us the key to propose the mathematical object of the symbolic computation system in next section.

4   Symbolic Computation Tool

In this section, we follow F. San–Juan and A. Abad [20] to introduce the representation of a mathematical object in a computer. To that purpose, let us introduce the concepts of normal and canonical functions. Let E be a set of symbolic objects, and $\sim$ be an equivalence relation in E, defined as follows: $a \sim b$ if $a = b$, with $a, b \in E$. Here, the operator $=$ is considered as the equality on the mathematical object level. Moreover, $a \equiv b$ if a and b are identical as symbolic objects. A function $f : E \to E$ is said to be normal in $(E, \sim)$ if $f(a) \sim a$ for all $a \in E$, and f is said to be canonical in $(E, \sim)$ if it is normal and $a \sim b \Rightarrow f(a) = f(b)$ for all $a, b \in E$. Thus, a canonical function provides identical objects when objects are equivalent, that is, when they represent the same mathematical object.

A general implementation of the Poincaré–Lindstedt technique should be constructed over a kernel which works with series of the form

$$S(T) = \sum_{m \in \mathbb{I}} P_m(A_1, A_2, \tau_1, \ldots, \tau_n) \sin(mT) + Q_m(A_1, A_2, \tau_1, \ldots, \tau_n) \cos(mT), \qquad (26)$$

where $\mathbb{I} \subset \mathbb{N}$, and P_m and Q_m are polynomials of the form

$$P_m(A_1, A_2, \tau_1, \ldots, \tau_n) = \sum_{i,j} c_{i,j} A_1^{j_1} A_2^{j_2} \tau_1^{i_1} \cdots \tau_n^{i_n},$$

$$Q_m(A_1, A_2, \tau_1, \ldots, \tau_n) = \sum_{i',j'} c_{i',j'} A_1^{j'_1} A_2^{j'_2} \tau_1^{i'_1} \cdots \tau_n^{i'_n},$$

being $j_1, j_2, i_1, \ldots, i_n$ and $j'_1, j'_2, i'_1, \ldots, i'_n$ natural numbers, and $c_{i,j}, c_{i',j'}$ real and rational. That is, the mathematical object we need to apply the Poincaré–Lindstedt method is a particular type of a Poisson series. The term $A_1^{j_1} A_2^{j_2} \tau_1^{i_1} \cdots \tau_n^{i_n}$ is usually referred to as monomial, the letters $A_1, A_2, \tau_1, \ldots, \tau_n$ as the polynomial variables, and T as the angle variable.

The set of Poisson series forms a commutative algebra over the ring of coefficients [20]. If P and Q are Poisson series, then their sum and product by a real number, $P + Q$ and αP, with $\alpha \in \mathbb{R}$, are also Poisson series. Algebraic closure properties make automatic manipulation rather easy when the elements in the algebra are represented in a standard canonical form, since closure implies that the result retains the standard form of the operands.

Some of the specific symbolic operations we have to implement in order to apply the method are:

1. Computation of derivatives with respect to T.
2. Resolution of systems of linear differential equations of first order by the method of undetermined coefficients.
3. Symbolic solution of systems of equations.

We will focus our attention on the mathematical object defined by the special type of Poisson series given in (26). We will refer to the set of these Poisson series as $\mathbb{P}$. Now, we look for a canonical representation for each equivalence class defined in $\mathbb{P}$. For that purpose, the following operations must be performed over each series:

1. Let us consider a Poisson series

$$S(T) = \sum_{m \in \mathbb{I}} P_m(A_1, A_2, \tau_1, \ldots, \tau_n) \sin(mT) + Q_m(A_1, A_2, \tau_1, \ldots, \tau_n) \cos(mT).$$

If $m < 0$, the following rules must be applied:

$$\sin(-mT) = -\sin(mT), \quad \cos(-mT) = \cos(mT).$$

2. Let

$$P_m(A_1, A_2, \tau_1, \ldots, \tau_n) \sin(mT) + Q_m(A_1, A_2, \tau_1, \ldots, \tau_n) \cos(mT)$$

be a term of a Poisson series, where

$$P_m(A_1, A_2, \tau_1, \ldots, \tau_n) = \sum_{i,j} c_{i,j} A_1^{j_1} A_2^{j_2} \tau_1^{i_1} \cdots \tau_n^{i_n} ,$$

$$Q_m(A_1, A_2, \tau_1, \ldots, \tau_n) = \sum_{i',j'} c_{i',j'} A_1^{j_1'} A_2^{j_2'} \tau_1^{i_1'} \cdots \tau_n^{i_n'} .$$

The terms of P_m (and Q_m) must be ordered as follows: let us condider two terms of P_m,

$$T_1 = c_{i,j} A_1^{j_1} A_2^{j_2} \tau_1^{i_1} \cdots \tau_n^{i_n} ,$$

and

$$T_2 = c_{i',j'} A_1^{j_1'} A_2^{j_2'} \tau_1^{i_1'} \cdots \tau_n^{i_n'} .$$

We say that $T_1 < T_2$ if $(j_1 < j_1')$ or $(j_1 = j_1'$ and $j_2 < j_2')$ or $(j_1 = j_1'$, $j_2 = j_2'$ and for the first $\nu \in \{1, \ldots, n\}$ such that $i_\nu \neq i_\nu'$, then $i_\nu < i_\nu'$). If $j_\alpha = j_\alpha'$ and $i_\nu = i_\nu'$ for each $\alpha = 1, 2$ and $\nu = 1, \ldots, n$, then the terms must be grouped together.

Now, we will consider the special Poisson series set we are working with from the computational point of view. To that purpose, we will analyze the basic information which characterizes a Poisson series, as well as the data structure to store it in the computer. This must be done preserving the canonical representation we have chosen.

The efficiency of the algorithms for the basic algebra of a series depends on the way is coded. An overcoded structure that makes good use of memory generally requires complex algorithms, which increase the computational cost in terms of time. On the other hand, an undercoded computational representation of the series generates simple algorithms, because the location of all the coefficients can be obtained directly. However, this scheme presents the inconvenience of being very wasteful in the memory resources required for the storage of the series [7]. As pointed out in [20], most of the operations involving a series are based on navigating and searching through the structure that represents the series. Thus, and taking into account the special type of Poisson series we are dealing with, the most adequate structure for storing these series is a linked and ordered list, where each node of the structure is linked to two red–black trees for the storage of the polynomials P_m and Q_m respectively. We show a representation of this structure in Figure 2.
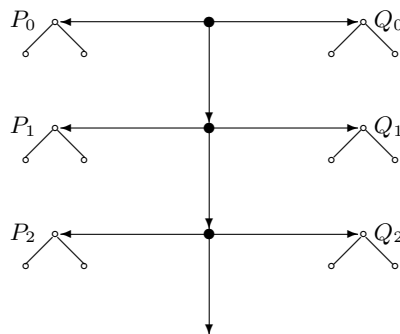

Figure 2. A dynamical list represents a Poisson series.

Now, we will concentrate on the way the polynomial parts P_m and Q_m are stored. To this purpose, let us introduce the red–black tree structure. A red–black tree is a special type of tree, where each node has a color attribute, the value of which is either red or black. In addition to the ordinary requirements imposed on binary search trees, the following additional requirements of any valid red-black tree apply: A

node is either red or black. The root is black. All leaves are black, even when the parent is black. Both children of every red node are black. Every simple path from a node to a descendant leaf contains the same number of black nodes. A critical property of red–black trees is enforced by these constraints: the longest path from the root to a leaf is no more than twice as long as the shortest path from the root to a leaf in that tree. The result is that the tree is roughly balanced. Since operations such as inserting, deleting, and finding values requires worst–case time proportional to the height of the tree, this fact makes the red–black tree efficient. For instance, the search–time is $O(\log n)$.

As explained above, each node of the Poisson series structure is linked with two red–black trees that represent P_m and Q_m respectively. Let us now consider one of these two polynomials,

$$P_m(A_1, A_2, \tau_1, \ldots, \tau_n) = \sum_{i,j} c_{i,j} A_1^{j_1} A_2^{j_2} \tau_1^{i_1} \cdots \tau_n^{i_n} \,.$$

The information associated to each term of P_m is given by the following elements:

1. A real number $c_{ij} \in \mathbb{R}$.
2. A set of $n+2$ integer numbers, $j_1, j_2, i_1 \ldots, i_n$.

The data associated to each node of the tree is a real number representing the coefficient of the corresponding term ($c_{i,j}$), and the key of each node is given by the set $(j_1, j_2, i_1 \ldots, i_n)$. In Figure 3, we show the tree structure in which a polynomial is stored. The polynomial we show in Figure 3 has 7 terms, with keys $\nu_1 < \nu_2 < \cdots < \nu_7$. Each key corresponds to a set of numbers of the form
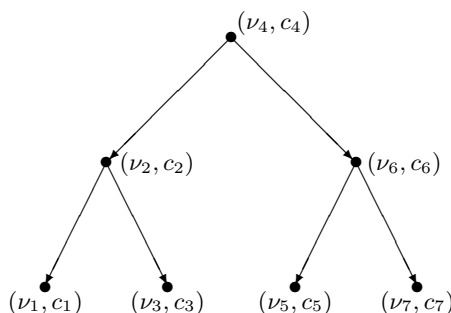
$$\nu = (j_1, j_2, i_1, \ldots, i_n) \,.$$

Figure 3. Red–black tree for representing a polynomial with keys $\nu_1 < \nu_2 < \nu_3 < \nu_4 < \nu_5 < \nu_6 < \nu_7$. Each node of the tree contains also the value of the coeffcient of the term (c_i).

If we store the key of a term in a vector structure, the complexity of the comparison of the keys is $O(n)$. We can reduce this complexity by storing keys in red–black trees. For each term of a Poisson series, we store pairs (ν, i_ν). Thus, the complexity of comparison between terms is reduced from $O(n)$ to $O(\log_2(n))$ in the worst case scenario. If the keys associated to two different terms have different sizes, that means both terms are not equal and can not be collected. This fact helps also to reduce the computation time. Moreover, it is not necessary to compare the entire key in case one index fails.

Thus, from a computational point of view, a Poisson series will be represented by a red–black tree with keys stored in red–black trees. In Figure 4, we show the representation of the Poisson series

$$\begin{aligned} P = \quad &\left(A_1^3 + A_1^2 A_2 + A_2^3\right)\sin T + \left(A_1^3 + A_1 A_2^2 - 2A_2^3\right)\cos T + \\ &+ \left(A_1^3 + 3A_1^2 A_2 + 2A_2^3\right)\sin 3T - \left(A_1^3 - 3A_1 A_2^2 + 7A_2^3\right)\cos 3T \,, \end{aligned}$$

just to clarify the way red–black trees are used to store a Poisson series. We also illustrate the way the key is coded in a red–black tree structure. In Figure 5, we show the representation of the key $A_1^2 A_2^4 \tau_1^7$.
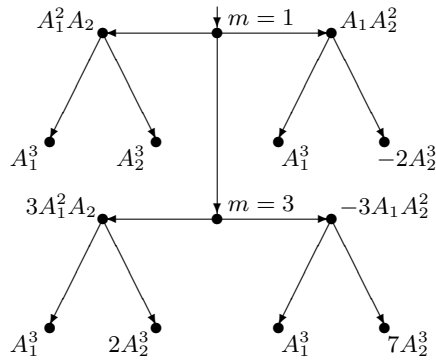
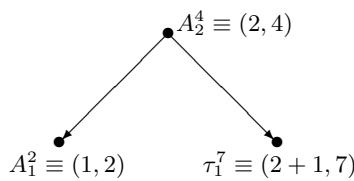**Figure 4.** Representation of a Poisson series.

Figure 5. Representation of the key $A_1^2 A_2^4 \tau_1^7$ in a red–black tree.

5   Symbolic Algorithm for the Poincaré–Lindstedt Method

In this section, we present a general algorithm to compute periodic solutions through the application of the Poincaré–Lindstedt method to the system given in equation (1). As above, $\mathbb{P}$ refers to the set of Poisson series of the type defined in (26).

1. Define $X(\rho_1, \rho_2, q) \in \mathbb{P}$ for each $\rho_1, \rho_2, q \in \mathbb{N}$ such that $0 \le \rho_1, \rho_2 \le M$ and $0 \le q \le Q$, Q being the order of the expansion. Here,
$$X(\rho_1, \rho_2, q) = (x^{\rho_1} y^{\rho_2})_q \,.$$

2. Define $DX(q), DY(q) \in \mathbb{P}$ for each $q \in \mathbb{N}$ such that $0 \le q \le Q$, Q being the order of the expansion. Here,
$$DX(q) = \frac{d}{dT} x_q \,, \quad DY(q) = \frac{d}{dT} y_q \,.$$

3. Define the array $W(q) \in \mathbb{P}$, for each $0 \le q \le Q$, to represent the coefficient ω_q.
4. The functions $f(x,y)$ and $g(x,y)$ are represented by the following $(1+M) \times (1+M)$ real matrices,

$$F = \begin{pmatrix} f_{0,0} & \cdots & f_{0,M} \\ \vdots & \ddots & \vdots \\ f_{M,0} & \cdots & f_{M,M} \end{pmatrix},$$

$$G = \begin{pmatrix} g_{0,0} & \cdots & g_{0,M} \\ \vdots & \ddots & \vdots \\ g_{M,0} & \cdots & g_{M,M} \end{pmatrix}.$$

If $f(x,y)$ and $g(x,y)$ are given by equation (2), then $f_{\rho_1,\rho_2} = 0$ and $g_{\rho_1,\rho_2} = 0$ if $\rho_1 + \rho_2 > M$. We will refer to the element f_{ρ_1,ρ_2} and g_{ρ_1,ρ_2} as $F(\rho_1, \rho_2)$ and $G(\rho_1, \rho_2)$ respectively.

5.1 0–th Order Solution

1. Set $X(0,0,0) = 1$ and $W(0) = 1$.
2. Set $W(0) = 1$. This means that $\omega_0 = 1$.
3. Compute $X(1,0,0)$ and $X(0,1,0)$ as the solution to (6). Notice that $X(1,0,0)$ and $X(0,1,0)$ are modified Poisson series containing parameters with undetermined value corresponding to the integration constants.
4. Compute

$$DX(0) = \frac{d}{dt}X(1,0,0)\,,$$
$$DY(0) = \frac{d}{dt}X(0,1,0)\,.$$

5. Calculate, for each ρ with $2 \leq \rho \leq M$,

$$X(\rho,0,0) = X(1,0,0)X(\rho-1,0,0)\,,$$
$$X(0,\rho,0) = X(0,1,0)X(0,\rho-1,0)\,.$$

that is, x_0^ρ and y_0^ρ.
6. For each ρ_1, ρ_2 with $1 \leq \rho_1, \rho_2 \leq M$, compute the modified Poisson series $(x^{\rho_1}y^{\rho_2})_0$,

$$X(\rho_1,\rho_2,0) = X(\rho_1,0,0)X(0,\rho_2,0)\,.$$

5.2 1–th Order Solution

1. Compute the first term of the right–hand side of equation (7),

$$U_1 = \sum_{0 \leq q \leq M} \sum_{0 \leq \nu \leq q} f_{\nu,q-\nu} x_0^\nu y_0^{q-\nu}\,,$$
$$V_1 = \sum_{0 \leq q \leq M} \sum_{0 \leq \nu \leq q} g_{\nu,q-\nu} x_0^\nu y_0^{q-\nu}\,.$$

These series are calculated as

$$U_1 = \sum_{0 \leq q \leq M} \sum_{0 \leq \nu \leq q} F(\nu,q-\nu)X(\nu,q-\nu,0)\,,$$
$$V_1 = \sum_{0 \leq q \leq M} \sum_{0 \leq \nu \leq q} G(\nu,q-\nu)X(\nu,q-\nu,0)\,.$$

2. Calculate the rest of the right–hand side of (7),

$$R_1 = -\omega_1 x_0'\,, \quad S_1 = -\omega_1 y_0'\,.$$

The value of ω_1 is fitted to assure that no secular terms are included in the solution. To do that, we apply the following result: the system

$$x' + y = A \sin T + B \cos T + \sum_{m \geq 2} A_m \sin(mT) + B_m \cos(mT)\,,$$
$$y' - x = C \sin T + D \cos T + \sum_{m \geq 2} C_m \sin(mT) + D_m \cos(mT)\,,$$

has periodic solutions if, and only if, $A - D = 0$ and $B + C = 0$.
3. Once $W(1)$ has been determined, we substitute its value in equation (8). This is equivalent to eliminating resonant terms from the right–hand side of this equation. Now, we calculate $X(1,0,1)$ and $X(0,1,1)$ as the solution to system (8) without resonant terms.

4. Compute

$$DX(1) = \frac{d}{dt}X(1,0,1)\,,$$

$$DY(1) = \frac{d}{dt}X(0,1,1)\,,$$

that is, x_1' and y_1'.

5. Calculate, for each ρ with $2 \leq \rho \leq M$,

$$X(\rho,0,1) = \sum_{0 \leq \nu \leq 1} X(\rho-1,0,\nu)X(1,0,1-\nu)\,,$$

$$X(0,\rho,1) = \sum_{0 \leq \nu \leq 1} X(0,\rho-1,\nu)X(0,1,1-\nu)\,,$$

that is, $(x^\rho)_1$ and $(y^\rho)_1$.

6. For each ρ_1, ρ_2 with $1 \leq \rho_1, \rho_2 \leq M$, compute $(x^{\rho_1}y^{\rho_2})_1$ as

$$X(\rho_1,\rho_2,1) = \sum_{0 \leq \nu \leq 1} X(\rho_1,0,\nu)X(0,\rho_2,1-\nu)\,.$$

5.3 p–th Order Solution, for $p > 1$

1. Compute the following part of the right–hand side of equation (8),

$$U_p = \sum_{0 \leq q \leq M} \sum_{0 \leq \nu \leq q} f_{\nu,q-\nu}\left[x^\nu y^{q-\nu}\right]_{p-1} - \sum_{1 \leq \nu \leq p-1} x_\nu' \omega_{p-\nu}\,,$$

$$V_p = \sum_{0 \leq q \leq M} \sum_{0 \leq \nu \leq q} g_{\nu,q-\nu}\left[x^\nu y^{q-\nu}\right]_{p-1} - \sum_{1 \leq \nu \leq p-1} y_\nu' \omega_{p-\nu}\,.$$

which corresponds to

$$U_p = \sum_{0 \leq q \leq M} \sum_{0 \leq \nu \leq q} F(\nu,q-\nu)X(\nu,q-\nu,p-1) - \sum_{1 \leq \nu \leq p-1} DX(\nu)W(p-\nu)\,,$$

$$V_p = \sum_{0 \leq q \leq M} \sum_{0 \leq \nu \leq q} G(\nu,q-\nu)X(\nu,q-\nu,p-1) - \sum_{1 \leq \nu \leq p-1} DY(\nu)W(p-\nu)\,.$$

2. Calculate the rest of the right–hand side of (7),

$$R_p = -\omega_p x_0'\,, \quad S_p = -\omega_p y_0'\,.$$

The value of ω_p is fitted to assure that resonance disappears. As before, we apply the following result: the system

$$x' + y = A\sin T + B\cos T + \sum_{m \geq 2} A_m \sin(mT) + B_m \cos(mT)\,,$$

$$y' - x = C\sin T + D\cos T + \sum_{m \geq 2} C_m \sin(mT) + D_m \cos(mT)\,,$$

has periodic solutions if, and only fi, $A - D = 0$ and $B + C = 0$. Then ω_p must be fitted to make resonance disappear, and we can compute $W(p)$ as a modified Poisson series depending on the integration constants of the problem.

3. Once $W(p)$ has been determined, we substitute it in equation (8). This corresponds to eliminating resonant terms from the right–hand side of this equation. Now, we calculate $X(1,0,p)$ and $X(0,1,p)$ as the solution to system (8) without resonant terms. For that purpose we use the undetermined coefficients method.

4. Compute

$$DX(p) = \frac{d}{dt}X(1, 0, p),$$

$$DY(p) = \frac{d}{dt}X(0, 1, p).$$

5. For each ρ with $2 \le \rho \le M$, determine

$$X(\rho, 0, p) = \sum_{0 \le \nu \le p} X(\rho - 1, 0, \nu)X(1, 0, p - \nu),$$

$$X(0, \rho, p) = \sum_{0 \le \nu \le p} X(0, \rho - 1, \nu)X(0, 1, p - \nu),$$

that is $(x^\rho)_p$ and $(y^\rho)_p$.

6. For each ρ_1, ρ_2 with $1 \le \rho_1, \rho_2 \le M$, compute $(x^{\rho_1}y^{\rho_2})_p$,

$$X(\rho_1, \rho_2, p) = \sum_{0 \le \nu \le p} X(\rho_1, 0, \nu)X(0, \rho_2, p - \nu).$$

The approximation to the periodic solution to (1) is given by

$$x(t) = X(1, 0, 0)(T) + \epsilon X(1, 0, 1)(T) + \epsilon^2 X(1, 0, 2)(T) + \cdots + \epsilon^Q X(1, 0, Q),$$
$$y(t) = X(0, 1, 0)(T) + \epsilon X(0, 1, 1)(T) + \epsilon^2 X(0, 1, 2)(T) + \cdots + \epsilon^Q X(0, 1, Q),$$

where $X(1, 0, q)(T)$ and $X(0, 1, q)$ $(1 \le q \le Q)$ are 2π–periodic in $T = \omega t$, and

$$\omega = W(0) + \epsilon W(1) + \epsilon^2 W(2) + \cdots + \epsilon^Q W(Q).$$

6   Conclusion

We have presented a new symbolic computation tool to implement the algorithm of Poincaré–Lindstedt method for the computation of periodic solutions in two dimensional differential systems of first order. After giving a general review of the method, we have applied it to a Lotka–Volterra system in order to understand the mathematical object we deal with when we apply the Poincaré–Lindstedt method. A general implementation of the Poincaré–Lindstedt technique should be constructed over a kernel which works with a particular type of a Poisson series. We have also introduced a computational representation of these series, using red–black trees within which keys are stored. Finally, we have presented the algorithm to implement the Poincaré–Lindstedt method.

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