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On Some Limit Properties for $(a_n, \phi(n))$-Asymptotic Circular Markov Chains

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Abstract The purpose of this article is to obtain some limit properties for $(a_n, \phi(n))$-Asymptotic Circular Markov Chains. This paper firstly presents some limit theorems of delayed sums for finite $(a_n, \phi(n))$-Asymptotic Circular Markov Chains and then establishes the generalized Shannon-McMillan-Breiman theorem [1, 2, 3, 4].

Keywords: $(a_n, \phi(n))$-asymptotic circular Markov Chain, delayed sum, generalized Shannon-McMillan theorem.

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space and $(\xi_n)_{n=0}^{\infty}$ be a nonhomogeneous Markov chain taking values in $\mathcal{X} = \{1, 2, \cdots, b\}$ with the transition matrices,

$$P_n = (p_n(i, j)), i, j \in \mathcal{X}, n \geq 1, \tag{1}$$

where $p_n(i, j) = P(\xi_n = j|\xi_{n-1} = i)$. For simplicity, $\xi_{m,n}$ represents the random vector of $(\xi_m, \xi_{m+1}, \cdots, \xi_{m+n})$ and $x_{m,n} = (x_m, x_{m+1}, \cdots, x_{m+n})$, a realization of $\xi_{m,n}$ Let the joint distribution of $\xi_{m,n}$ be

$$p(x_{m,n}) = P(\xi_{m,n} = x_{m,n}), \quad x_k \in \mathcal{X}, \quad m \leq k \leq m + n \tag{2}$$

Let $f^{(0)}$ be a probability distribution on $\mathcal{X}$ and let

$$P^{(m,n)} := P_{m+1}P_{m+2} \cdots P_n, \tag{3}$$

$$f^{(k)} := f^{(0)}P_1P_2 \cdots P_k. \tag{4}$$

For convenience, let $p^{(m,n)}(i, j)$ denote the (i, j) element of $P^{(m,n)}$ and $f^{(k)}(j)$ be the j th element of $f^{(k)}$. It is easy to see that

$$p^{(m,n)}(i, j) = P(\xi_n = j|\xi_m = i), \tag{5}$$

$$f^{(k)}(j) = P(\xi_k = j). \tag{6}$$

If the Markov chain is homogeneous, then $\{P_n, n \geq 1\}$ will be denoted simply by P and $P^{(m,m+k)}$ is P^k.

Let $A = (a_{ij})$ be a matrix on $\mathcal{X} \times \mathcal{X}$. We define the norm $\|\cdot\|$ of A as follows:

$$\|A\| := \sup_{i\in\mathcal{X}} \sum_{j\in\mathcal{X}} |a_{ij}|. \tag{7}$$

If $f = (f_1, f_2, \cdots)$ is a row vector, we define $\|f\| = \sum_{j=1}^{\infty} |f_j|$, if $g = (g_1, g_2, \cdots)^T$ is a column vector, we define $\|g\| = \sup_{i\in\mathcal{X}} |g_i|$. The norms defined as above satisfy the following properties:
(a) $\|AB\| \leq \|A\| \cdot \|B\|$ for all matrices A and B;
(b) $\|P\| = 1$ for any stochastic matrix P.
These two properties will be used repeatedly in this article.

Definition 1. Let Q be a "constant" stochastic matrix (i.e. Q is a stochastic matrix each row of which is the same). The sequence $\{P_n, n \geq 1\}$ is said to be strongly ergodic (with constant stochastic matrix Q) if for every $m \geq 0$

$$\lim_{n\to\infty} \|P^{(m,n)} - Q\| = 0. \tag{8}$$

Throughout this paper we always assume that $(a_n)_{n=0}^{\infty}$ and $(\phi_n)_{n=0}^{\infty}$ are two sequences of nonnegative integers such that $\phi(n)$ tends to infinity as $n \to \infty$.

The sequence $\{P_n, n \geq 1\}$ is called to be $(a_n, \phi(n))$-strong ergodicity of Markov chains in the Cesáro sense (to constant stochastic matrix Q) if for every $m \geq 0$

$$\lim_{n\to\infty} \left\| \frac{1}{\phi(n)} \sum_{t=a_n+1}^{a_n+\phi(n)} P^{(m,t)} - Q \right\| = 0. \tag{9}$$

If the Markov chain is homogeneous, (9) become

$$\lim_{n\to\infty} \left\| \frac{1}{\phi(n)} \sum_{t=a_n+1}^{a_n+\phi(n)} P^{t} - Q \right\| = 0. \tag{10}$$

An irreducible stochastic matrix P, of period $d(d \geq 1)$ partitions the state space $\mathcal{X}$ into d disjoint subspaces $C_0, C_1, \cdots, C_{d-1}$, and P_d yields d stochastic matrices $\{T_l, 0 \leq l \leq d-1\}$, where T_l is defined on C_l . If the irreducible periodic stochastic matrix P is finite, then each T_l is automatically strongly ergodic, but if the irreducible periodic stochastic matrix P is infinite, the strong ergodicity of T_l is not guaranteed. If each T_l is strongly ergodic, then the stochastic matrix will be called periodic strongly ergodic [5].

Definition 2. Let $(\xi_n)_{n=0}^{\infty}$ be a non-homogeneous Markov Chain with the initial distribution $f^{(0)}$ and the transition matrices of (1.1), $T_l = (t_l(i,j)), (l = 1, 2, \cdots d)$ be d transition matrices. The following Markov chains is called an $(a_n, \phi(n))$-asymptotic circular Markov chain of moving average if

$$\lim_{n\to\infty} \frac{1}{\phi(n)} \sum_{t=a_n+1}^{a_n+\phi(n)} \|P_{td+l} - T_l\| = 0, \quad l = 1, 2, \cdots, d. \tag{11}$$

In particular, if $a_n = 0, \phi(n) = n$ and

$$P_{td+l} = T_l, \quad l = 1, 2, \cdots, d, \quad t = 0, 1, 2, \cdots \tag{12}$$

this Markov chain is called a circular Markov chain.

Circular Markov chains play an important role in the nearest neighbour random walks on inhomogeneous Markov chains periodic lattices, i.e[6]. periodic lattices which consist of a periodically repeated unit cell, where unit cell contains a number of non-equivalent sites [7], an example of circular Markov process occurs in the evaluation system of M/M/C queueing system in which the servers, after an idle period, only restart work when enough customers arrived to the system.

Definition 3. Let $(\xi_n)_{n=0}^{\infty}$ be a nonhomogeneous Markov chain taking on values in $\mathcal{X}$. Let

$$f_{a_n,\phi(n)}(\omega) = -\frac{1}{\phi(n)} \log p(\xi_{a_n,\phi(n)}), \tag{13}$$

where log is the natural logarithm. $f_{a_n,\phi(n)}(\omega)$ is called generalized entropy density of $\xi_{a_n,\phi(n)}$ and $f_{a_n,\phi(n)}(\omega)$ can be rewritten as

$$f_{a_n,\phi(n)}(\omega) = -\frac{1}{\phi(n)} \left\{ \log f^{(a_n)}(\xi_{a_n}) + \sum_{k=a_n+1}^{a_n+\phi(n)} \log p_k(\xi_{k-1}, \xi_k) \right\} \tag{14}$$

Let Q be another prohability measure on $(\Omega, \mathcal{H},)$ if $(\eta_n)_{n=0}^{\infty}$ is a nonhomogeneous Markov chain under Q with initial distribution

$$(q(1), q(2), \ldots, q(b)), \tag{15}$$

and transition matrices

$$Q_n = (q_n(i,j))_{b\times b}, \quad i,j \in \mathcal{X}, n \geq 1, \tag{16}$$

where $q_n(i,j) = Q(\eta_n = j|\xi_{n-1} = i)$. Define

$$q(x_{a_n,\phi(n)}) = p(x_{a_n}) \prod_{k=a_n+1}^{a_n+\phi(n)} q_k(x_{k-1}, x_k). \tag{17}$$

$$\mathcal{L}_{a_n,\phi(n)}(\omega) = -\frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \log \frac{q_k(\eta_{k-1}, \eta_k)}{p_k(\xi_{k-1}, \xi_k)} \tag{18}$$

$$\mathcal{L}(\omega) = \limsup_n \mathcal{L}_{a_n,\phi(n)}(\omega) = -\liminf_n \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \log \frac{q_k(\eta_{k-1}, \eta_k)}{p_k(\xi_{k-1}, \xi_k)} \tag{19}$$

$\mathcal{L}_{a_n,\phi(n)}(\omega)$ and $\mathcal{L}(\omega)$ are called generalized sample relative entropy and generalized sample relative entropy rate between P and Q respectively.

Our paper is aim at extending the known results, the approach used in this paper different from [8] and[9], the essence of the method is first to construct a one parameter class of random variables with means of 1, then, using Borel-Cantelli lemma, to prove the existence of a.e. convergence of certain random variables[10]. Under the condition of Lemma 1 of [10], we first give the definition of an $(a_n, \phi(n))$-asymptotic circular Markov chain and prove some lemmas. Then, we prove generalized limit theorems for countable $(a_n, \phi(n))$-asymptotic circular Markov chains, as corollaries, we obtain the strong law of large number for non-homogeneous Markov chains which is known results of [10] as well as extending the results of [8], Finally, we achieve the generalized Shannon-McMillan Breiman theorem for finite $(a_n, \phi(n))$-asymptotic circular Markov chains, which is, to some extent, an extension of the result of[9].

The remaining paper is organized as follows. Section 2 provides some related Lemmas. Section 3 gives the main results and the proofs.

2 Some Lemmas

Before proving the main results, we firstly prove some related lemmas, which will play important roles in achieving our results.

Lemma 1. *[11] Let $Q_l(l = 1, 2, \cdots, d)$ be d stochastic matrices and let $R_1 = Q_1Q_2\cdots Q_d, R_2 = Q_2Q_3\cdots Q_dQ_1$,*
$\cdots, R_d = Q_dQ_1\cdots Q_{d-1}$. If R_1 is $(a_n, \phi(n))$-strongly ergodic with constant stochastic matrix T_1. Then $R_2, R_3, \cdots, R_d$ are also $(a_n, \phi(n))$-strongly ergodic with the constant stochastic matrices $T_2, T_3, \cdots, T_d$, resp., where $T_l = T_1 \prod_{i=1}^{l-1} Q_i (l = 2, \cdots, d)$.

Proof. Since T_1 is a constant stochastic matrix, which implies that $T_l(l = 2, \cdots, d)$ are also constant stochastic matrices. For any stochastic matrix P and a constant stochastic matrix Q we have $PQ = Q$. Since R_1 is $(a_n, \phi(n))$-strongly ergodic with a constant stochastic matrix T_1, then

$$\lim_{n\to\infty} \|\frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} R_1^k - T_1\| = 0. \tag{20}$$

when $l \geq 2$, note that

$$\begin{aligned}\frac{1}{\phi(n)-1}\sum_{k=a_n+2}^{a_n+\phi(n)} R_l^k - T_l =&\frac{1}{\phi(n)-1}\sum_{k=a_n+2}^{a_n+\phi(n)} (R_l^k - T_l)\\ =&\frac{1}{\phi(n)-1}\sum_{k=a_n+2}^{a_n+\phi(n)}\left[\left(\prod_{i=1}^{d} Q_i\right) R_1^{k-1}\left(\prod_{i=1}^{l-1} Q_i\right) - \left(\prod_{i=1}^{d} Q_i\right) T_1 \left(\prod_{i=1}^{l-1} Q_i\right)\right]\\ =&\left(\prod_{i=1}^{d} Q_i\right)\left(\frac{1}{\phi(n)-1}\sum_{k=a_n+2}^{a_n+\phi(n)} R_1^{k-1} - T_1\right)\left(\prod_{i=1}^{l-1} Q_i\right)\end{aligned} \tag{21}$$

we have

$$\|\frac{1}{\phi(n)-1}\sum_{k=a_n+2}^{a_n+\phi(n)} R_l^k - T_l\| \leq \|\frac{1}{\phi(n)-1}\sum_{k=a_n+2}^{a_n+\phi(n)} R_1^{k-1} - T_1\| \tag{22}$$

(20) and (22) imply that

$$\lim_{n\to\infty} \|\frac{1}{\phi(n)}\sum_{k=a_n+1}^{a_n+\phi(n)} R_l^k - T_l\| = 0. \tag{23}$$

This means $R_l (l = 2, \cdots, d)$ are $(a_n, \phi(n))$-strongly ergodic. □

Lemma 2. *Let $Q_l (l = 1, 2, \cdots, d)$ be d stochastic matrices. Let $\{Q_n, n \geq 1\}$ be a sequence of stochastic matrices satisfying*

$$Q_{td+l} = Q_l, \quad l = 1, 2, \cdots, d, \quad t = 0, 1, 2\cdots \tag{24}$$

Let $P^{(m,n)}$ be defined as in (3). If (11) holds, then, for any positive integer k,

$$\frac{1}{\phi(n)}\sum_{k=a_n+1}^{a_n+\phi(n)} \|P^{(td+l,td+l+k)} - Q^{(td+l,td+l+k)}\| = 0, \quad l = 1, 2, \cdots, d. \tag{25}$$

Proof. For $k = 2$, we have by (24)

$$\begin{aligned}&\|P^{(td+l,td+l+2)} - Q^{(td+l,td+l+2)}\|\\ =&\|P_{td+l+1}P_{td+l+2} - Q_{l+1}Q_{l+2}\|\\ =&\|P_{td+l+1}P_{td+l+2} - Q_{l+1}P_{td+l+2} + Q_{l+1}P_{td+l+2} - Q_{l+1}Q_{l+2}\|\\ \leq&\|P_{td+l+1} - Q_{l+1}\| + \|P_{td+l+2} - Q_{l+2}\|\end{aligned} \tag{26}$$

By (11), we have

$$\frac{1}{\phi(n)}\sum_{t=a_n+1}^{a_n+\phi(n)} \|P^{(td+l,td+l+2)} - Q^{(td+l,td+l+2)}\| = 0, \quad l = 1, 2, \cdots, d. \tag{27}$$

Similarly, for $k > 2$, (25) holds by induction. □

Lemma 3. *[10]Assume that $(\xi_n)_{n=0}^{\infty}$ is a nonhomogeneous Markov chain taking values in $\mathcal{X} = \{1, 2, \cdots, b\}$ with initial distribution (15) and the transition matrices as (16). Let $(g_n(x, y))_{n=0}^{\infty}$ be a sequence of real functions defined on $\mathcal{X} \times \mathcal{X}$. If for every $\varepsilon > 0$*

$$\sum_{n=1}^{\infty} \exp[-\varepsilon\phi(n)] < \infty, \tag{28}$$

and there exists a real number $0 < \gamma < \infty$ *such that*

$$\limsup_{n} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} E[|g_k(\xi_{k-1}, \xi_k)|^2 \exp(\gamma |g_k(\xi_{k-1}, \xi_k)|\xi_{k-1}] = c(\gamma; \omega) < \infty \quad a.e., \tag{29}$$

then, we have

$$\lim_{n} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \{g_k(\xi_{k-1}, \xi_k) - E[g_k(\xi_{k-1}, \xi_k)|\xi_{k-1}]\} = 0 \quad a.e. \tag{30}$$

Proof. See Lemma 1 of [10]. □

Lemma 4. *Let* $\{t_k\}_{k=0}^{\infty}$ *be a bounded sequence of points in the plane,* $\|t_k\| \leq M$*,* δ *be a positive number, and let* $N_n(\delta)$ *be the number of terms which not belong to* $U(0, \delta)$ *in the first n terms of the sequence. Then*

$$\lim_{n} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} t_k = 0 \tag{31}$$

holds if and only if

$$\lim_{n} \frac{1}{\phi(n)} N_n(\delta) = 0, \quad \forall \delta > 0. \tag{32}$$

Lemma 5. *Let* $\varphi(x)$ *be a bounded function defined on at area D, a be a interior point in D, and* $\varphi(x)$ *be continues at* $x = t$*, and let* $\{t_k, k \geq 1\}$ *be a collection of points in D. If*

$$\lim_{n} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \|t_k - t\| = 0$$

holds, then

$$\lim_{n} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \|\varphi(t_k) - \varphi(t)\| = 0. \tag{33}$$

Proof. By the continuity of the function, $\forall \varepsilon > 0, \exists \delta > 0$ satisfying $U(t, \delta) \subset D$, whenever $t_1 \in U(t, \delta)$, we have $|\varphi(t_1) - \varphi(t)| \leq \varepsilon$. Let $N_n(\delta)$ be the number of terms which not belong to $U(0, \delta)$ in the first n terms of sequence $\{|t_k - t|, k \geq 1\}$, and $M_n(\varepsilon)$ be the number of terms which are greater than ε in the first n terms of sequence $\{|\varphi(t_k) - \varphi(t)|\}_{k=1}^{\infty}$. Then,

$$M_n(\varepsilon) \leq N_n(\delta) \tag{34}$$

It follows from (1.11), Lemma 1, and (2.15) that

$$\lim_{n} \frac{1}{\phi(n)} M_n(\varepsilon) = 0, \quad \forall \varepsilon > 0. \tag{35}$$

Since the sequence $\{|\varphi(t_k) - \varphi(t)|\}_{k=1}^{\infty}$ is bounded, (2.14) follows from (2.16). □

Lemma 6. *Let* $(\xi_n)_{n=0}^{\infty}$ *be a nonhomogeneous Markov chain with initial distribution* (2) *and transition matrices* (16) *under measure P,* $(\eta_n)_{n=0}^{\infty}$ *also be a nonhomogeneous Markov chain with initial distribution* (15) *and transition matrices*(16)

$$Q_n = (q_n(i,j)), \quad q_n(i,j) > \tau, \quad 0 < \tau < 1, \quad i, j \in \mathcal{X}, n \geq 0, \tag{36}$$

then

$$\lim_{n} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \left\{ \log \frac{p_k(\xi_{k-1}, \xi_k)}{q_k(\xi_{k-1}, \xi_k)} - \sum_{j=1}^{b} p_k(\xi_{k-1}, j) \log \frac{p_k(\xi_{k-1}, j)}{q_k(\xi_{k-1}, j)} \right\} = 0, \ P - a.e. \tag{37}$$

Proof. Letting $g_k(s,t) = \log q_k(s,t)$ in Lemma 3, we have

$$\begin{aligned} E_P[|g_k(\xi_{k-1},\xi_k)|^2 \exp(\gamma|g_k(\xi_{k-1},\xi_k)|\xi_{k-1}] &= \sum_{j=1}^{b} (\log q_k(\xi_{k-1},j))^2 (q_k(\xi_{k-1},j))^\gamma p_k(\xi_{k-1},j) \\ &\le \sum_{j=1}^{b} (\log\tau)^2\tau^\gamma \le b(\log\tau)^2\tau^\gamma \end{aligned} \tag{38}$$

By Lemma 3, we can easily prove

$$\lim_n \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \left\{ \log \frac{p_k(\xi_{k-1},\xi_k)}{q_k(\xi_{k-1},\xi_k)} - \sum_{j=1}^{b} p_k(\xi_{k-1},j) \log \frac{p_k(\xi_{k-1},j)}{q_k(\xi_{k-1},j)} \right\} = 0, \ P-a.e. \tag{39}$$

□

3 The Main Results

In this section, we will present our main results based on previous Lemmas.

Theorem 1. *Let $(\xi_n)_{n=0}^\infty$ be an $(a_n, \phi(n))$-asymptotic circular Markov chain defined by Definition 1. Let $(g_n(x,y))_{n=0}^\infty$ be a sequence of real functions defined on $\mathcal{X}\times\mathcal{X}$. If, for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} \exp[-\varepsilon\phi(n)] < \infty, \tag{40}$$

and there exists a real number $0 < \gamma < \infty$ such that

$$\limsup_n \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} E[|g_k(\xi_{k-1},\xi_k)|^2 \exp(\gamma|g_k(\xi_{k-1},\xi_k)|\xi_{k-1}] = c(\gamma;\omega) < \infty \quad a.e. \tag{41}$$

Let

$$h_n(i) = \sum_{j\in\mathcal{X}} g_n(i,j) p_n(i,j) \tag{42}$$

h_n be a column vector with ith element $h_n(i)$ and $h^l(l = 1,2,\cdots,d)$ be d column vectors with ith elements $h^l(i)$. Let $R_l, T_l(l = 1,2,\cdots,d)$ be the same as in Lemma 2 and R_1 be $(a_n,\phi(n))$-strongly ergodic. If

$$\lim_{n\to\infty} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \|h_{td+l} - h^l\| = 0, \qquad l = 1,2,\cdots,d, \tag{43}$$

and $\|h_n\|$ and h^l are finite, then

$$\lim_{n\to\infty} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} g_k(\xi_{k-1},\xi_k) = \frac{1}{d} \sum_{i\in\mathcal{X}} \sum_{l=1}^{d} h^l(i)\pi^l(i) \quad a.e., \tag{44}$$

where $\pi^l = (\pi^l(1), \pi^l(2), \cdots)$ is the the common row vector of T_l and also the unique stationary distribution determined by $R_l(l = 1,2,\cdots,d)$.

Proof. By (40), (41) and Lemma 3, we have

$$\lim_n \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \left\{ g_k(\xi_{k-1},\xi_k) - E[g_k(\xi_{k-1},\xi_k)|\xi_{k-1}] \right\} = 0 \quad a.e. \tag{45}$$

Now, we consider

$$\begin{aligned}
&\frac{1}{\phi(n)}\sum_{k=1}^{\phi(n)} E[g_{a_n+k}(\xi_{a_n+k},\xi_{a_n+k+1})|\xi_{a_n}] \\
&= \frac{1}{\phi(n)}\sum_{k=1}^{\phi(n)}\sum_i\sum_j g_{a_n+k}(i,j)p_{a_n+k+1}(i,j)p^{(a_n,a_n+\phi(n))}(\xi_{a_n},i) \\
&= \frac{1}{\phi(n)}\sum_{k=1}^{\phi(n)}\sum_i h_{a_n+k}(i)p^{(a_n,a_n+\phi(n))}(\xi_{a_n},i) \qquad (46)\\
&= \frac{1}{\phi(n)}\sum_i\sum_{l=1}^{d}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1} h_{td+l+a_n}(i)p^{(td+l,td+l+a_n)}(\xi_{td+l},i) \\
&+ \frac{1}{\phi(n)}\sum_i\sum_{k=[\frac{\phi(n)}{d}]d+1}^{\phi(n)} h_{k+a_n}(i)p^{(k,k+a_n)}(\xi_k,i)
\end{aligned}$$

where $[\cdot]$ represents the greatest integer not more than x. The second term of (46) is defined to be zero if $\phi(n)/d$ is a positive integer. Obviously,

$$\frac{1}{\phi(n)}\sum_i\sum_{k=[\frac{\phi(n)}{d}]d+1}^{\phi(n)} g_{k+a_n}p^{(k,k+a_n)}(\xi_k,i) = 0 \qquad (47)$$

Let $\{Q_n, n \geq 1\}$ be the same as in Lemma 2. And $q^{(m,n)}(i,j)$ be the (i,j) element of $Q^{(m,n)}$, M be the upper bound of $\|g^l\|(l = 1, 2, \cdots, d)$. Let v be a positive integer, $a_n = vd$ and $h^{d+1} = h^1$. Since

$$\begin{aligned}
&\Bigg|\frac{1}{\phi(n)}\sum_i\sum_{l=1}^{d}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1} h_{td+l+vd+1}(i)p^{(td+l,td+l+vd)}(\xi_{td+l},i) \\
&- \frac{1}{\phi(n)}\sum_i\sum_{l=1}^{d}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1} h^{l+1}(i)q^{(td+l,td+l+a_n)}(\xi_{td+l},i)\Bigg| \\
&\leq \Bigg|\frac{1}{\phi(n)}\sum_i\sum_{l=1}^{d}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1} h_{td+l+vd+1}(i)p^{(td+l,td+l+vd)}(\xi_{td+l},i) \\
&- \frac{1}{\phi(n)}\sum_i\sum_{l=1}^{d}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1} h^{l+1}(i)p^{(td+l,td+l+vd)}(\xi_{td+l},i)\Bigg| \\
&+ \Bigg|\frac{1}{\phi(n)}\sum_i\sum_{l=1}^{d}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1} h^{l+1}(i)p^{(td+l,td+l+vd)}(\xi_{td+l},i) \\
&- \frac{1}{\phi(n)}\sum_i\sum_{l=1}^{d}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1} h^{l+1}(i)q^{(td+l,td+l+a_n)}(\xi_{td+l},i)\Bigg| \\
&\leq \frac{1}{\phi(n)}\sum_i\sum_{l=1}^{d}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1} \left|h_{td+l+vd+1}(i) - h^{l+1}(i)\right| p^{(td+l,td+l+vd)}(\xi_{td+l},i) \\
&+ \frac{1}{\phi(n)}\sum_i\sum_{l=1}^{d}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1} \left|h^{l+1}(i)\right| \cdot \left|p^{(td+l,td+l+vd)}(\xi_{td+l},i) - q^{(td+l,td+l+vd)}(\xi_{td+l},i)\right|
\end{aligned}$$

$$\leq \frac{1}{\phi(n)}\sum_{l=1}^{d}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1}\|h_{td+l+vd+1}-h^{l+1}\|+\frac{M}{\phi(n)}\sum_{l=1}^{d}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1}\|P^{(td+l,td+l+vd)}-Q^{(td+l,td+l+vd)}\| \quad (48)$$

It follows from (43) that the first term of (48) goes to zero when $n\to\infty$. By Lemma 2, the second term of (48) also tends to zero when $n\to\infty$. Combining (45)-(48), we have

$$\lim_{n\to\infty}\left\{\frac{1}{\phi(n)}\sum_{k=a_n+1}^{a_n+\phi(n)}g_k(\xi_{k-1},\xi_k)-\frac{1}{\phi(n)}\sum_i\sum_{l=1}^{d}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1}h^{l+1}(i)q^{(td+l,td+l+a_n)}(\xi_{td+l},i)\right\}=0 \quad a.e. \quad (49)$$

and for any positive integer N

$$\lim_{n\to\infty}\left\{\frac{1}{\phi(n)}\sum_{k=a_n+1}^{a_n+\phi(n)}g_k(\xi_{k-1},\xi_k)-\frac{1}{\phi(n)}\sum_i\frac{1}{N}\sum_{v=1}^{n}\sum_{l=1}^{d}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1}h^{l+1}(i)q^{(td+l,td+l+a_n)}(\xi_{td+l},i)\right\} \quad (50)$$
$$=0 \quad a.e.$$

Set $R_{d+1}=R_1, a_n=vd$. It is easy to see that for $l=1,2,\cdots,d$,

$$Q^{(td+l,td+l+vd)}=Q_{td+l+1}Q_{td+l+2}\cdots Q_{td+l+vd}=Q_{l+1}Q_{l+2}\cdots Q_{l+vd}=R_{l+1}^{v} \quad (51)$$

Let $r_{l+1}^{(v)}(i,j)$ be the v-step probabilities determined by R_{l+1}. Set $\pi^{d+1}=\pi_1$ and $T_{d+1}=T_1$, we have

$$\begin{aligned}
&\left|\frac{1}{\phi(n)}\sum_i\frac{1}{N}\sum_{v=1}^{n}\sum_{l=1}^{d}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1}h^{l+1}(i)q^{(td+l,td+l+vd)}(\xi_{td+l},i)-\sum_i\frac{1}{d}h^l(i)\pi^l(i)\right|\\
&=\left|\frac{1}{\phi(n)}\sum_i\frac{1}{N}\sum_{v=1}^{N}\sum_{l=1}^{d}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1}h^{l+1}(i)r_{l+1}^{(v)}(\xi_{td+l},i)-\sum_i\frac{1}{d}h^{l+1}(i)\pi^{l+1}(i)\right|\\
&\leq\sum_i\sum_{l=1}^{d}\left|h^{l+1}(i)\right|\cdot\left|\frac{1}{\phi(n)}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1}\frac{1}{N}\sum_{v=1}^{N}r_{l+1}^{(v)}(\xi_{td+l},i)-\frac{1}{d}\pi^{l+1}(i)\right|\\
&\leq\frac{1}{d}\sum_i\sum_{l=1}^{d}\left|h^{l+1}(i)\right|\cdot\left|\frac{d}{\phi(n)}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1}\frac{1}{N}\sum_{v=1}^{N}r_{l+1}^{(v)}(\xi_{td+l},i)-\frac{d}{\phi(n)}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1}\pi^{l+1}(i)\right|\\
&+\frac{1}{d}\sum_i\sum_{l=1}^{d}\left|h^{l+1}(i)\right|\cdot\left|\frac{d}{\phi(n)}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1}\pi^{l+1}(i)-\pi^{l+1}(i)\right|\\
&\leq\frac{1}{d}\sum_i\sum_{l=1}^{d}|h^{l+1}(i)|\cdot\frac{d}{\phi(n)}\sum_{t=0}^{[\frac{\phi(n)}{d}]-1}|\frac{1}{N}\sum_{v=1}^{N}r_{l+1}^{(v)}(\xi_{td+l},i)-\pi^{l+1}(i)|\\
&+\frac{1}{d}\sum_i\sum_{l=1}^{d}|h^{l+1}(i)|\pi^{l+1}(i)|\frac{d}{\phi(n)}[\frac{d}{\phi(n)}]-1|\\
&\leq\frac{1}{d}\sum_{l=1}^{d}\|h^{l+1}\|\|\frac{1}{N}\sum_{v=1}^{N}R_{l+1}^{v}-T_{l+1}\|+\frac{1}{d}\sum_{v=1}^{N}\|h^{l+1}\||\frac{d}{\phi(n)}[\frac{d}{\phi(n)}]-1|\\
&\leq M\frac{1}{d}\sum_{l=1}^{d}\|\frac{1}{N}\sum_{v=1}^{N}R_{l+1}^{v}-T_{l+1}\|+M|\frac{d}{\phi(n)}[\frac{d}{\phi(n)}]-1|.
\end{aligned} \quad (52)$$

Giving $\varepsilon>0$, by Lemma 1, we can choose a fixed N large enough so that the first term of (52) does not exceed ε. The second term of (52) tends to zero as n goes to infinity. By (50), (52) and the arbitrariness of ε, (44) follows. These complete the proof of Theorem 1. □

Corollary 1. *Let $(\xi_n)_{n=0}^{\infty}$ be an $(a_n, \phi(n))$-asymptotic circular Markov chain defined by Definition 2. Let $R_l(l = 1, 2, \cdots, d)$ be the same as in Lemma 1. Assume that R_1 is $(a_n, \phi(n))$-strongly ergodic. Let $g(x)$ be a bounded function defined on $\mathcal{X}$. Then*

$$\lim_{n\to\infty} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} g(\xi_k) = \sum_{i\in\mathcal{X}} \frac{1}{d} \sum_{l=1}^{d} g(i)\pi^l(i) \quad a.e., \tag{53}$$

Proof. Let $g_n(x, y) = g(x)$ in Theorem 1, then

$$h_{td+l}(i) = \sum_j g_{td+l}(i, j)p_{td+l}(i, j) = \sum_j g_{td+l}(i)p_{td+l}(i, j) = g(i) \tag{54}$$

Let $h^l(i) = \sum_j g(i)q_l(i, j) = g(i)$, where $q_l(i, j)$ is the (i, j) element of transition matrix Q_l , therefore (43) holds. Since $g(x)$ is bounded, thus $\|h_n\|$ and $\|h^l\|$ are finite and (41) also follows. Note that

$$\frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} g(\xi_{k-1}, \xi_k) = \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} g(\xi_{k-1}) \tag{55}$$

and

$$\sum_i \frac{1}{d} \sum_{l=1}^{d} h^l(i)\pi^l(i) = \sum_i \frac{1}{d} \sum_{l=1}^{d} g(i)\pi^l(i) \tag{56}$$

This corollary follows from Theorem 1 directly. □

Define the indicator function $\mathbf{1}_i(j)$ on $\mathcal{X}$ as follows:

$$\mathbf{1}_i(j) = \begin{cases} 1, \text{ if } j = i; \\ 0. \text{ if } j \neq i\ . \end{cases} \tag{57}$$

where $i = 1, 2, \cdots$

Corollary 2. *Let $(\xi_n)_{n=0}^{\infty}$ be an $(a_n, \phi(n))$-asymptotic circular Markov chain defined by Definition 2. Let $R_l(l = 1, 2, \cdots, d)$ be the same as in Lemma 1. Assume that R_1 is $(a_n, \phi(n))$-strongly ergodic. Let $S_{a_n,\phi(n)}(c,$ ÏL$)$ be the number of c in the sequence of $\xi_{a_n+1}(\omega), \xi_{a_n+2}(\omega), \cdots, \xi_{a_n+\phi(n)}(\omega)$, i.e. $S_{a_n,\phi(n)}(c, \omega) = \sum_{m=a_n+1}^{a_n+\phi(n)} \mathbf{1}_c(\xi_m)$. Then*

$$\frac{S_{a_n,\phi(n)}(c, \omega)}{\phi(n)} = \frac{1}{d} \sum_{l=1}^{d} \pi^l(c) \quad a.e., \tag{58}$$

Proof. Let $g(x) = \mathbf{1}_c(x)$ in Corollary 1. Obviously, $|g(x)| \le 1$. Noticing that

$$\frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} g(\xi_k) = \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \mathbf{1}_c(\xi_k) = \frac{S_{a_n,\phi(n)}(c, \omega)}{\phi(n)} \tag{59}$$

$$\sum_i \frac{1}{d} \sum_{l=1}^{d} g(i)\pi^l(i) = \sum_i \frac{1}{d} \sum_{l=1}^{d} \mathbf{1}_c(i)\pi^l(i) = \frac{1}{d} \sum_{l=1}^{d} \pi^l(c) \tag{60}$$

(3.19) follows from Corollary 1. □

Corollary 3. *Let $(\xi_n)_{n=0}^{\infty}$ be a non-homogeneous Markov chain . Let $\{g_n(x, y), n \ge 1\}$ and h_n be the same as in Theorem 1. Let P be a stochastic matrix and be periodic strongly ergodic. Let $h(i)$ be another function defined on $\mathcal{X}$, h be column vector with ith element $h(i)$. If conditions* (40) *and* (41) *hold resp., and*

$$\lim_{n\to\infty} \frac{1}{\phi(n)} \sum_{k=a_n+1}^{a_n+\phi(n)} \|P_k - P\| = 0 \tag{61}$$

$$\lim_{n\to\infty}\frac{1}{\phi(n)}\sum_{k=a_n+1}^{a_n+\phi(n)}\|h_k-h\|=0 \tag{62}$$

if $\|h_n\|$ *and* $\|g\|$ *are finite, then*

$$\lim_{n\to\infty}\frac{1}{\phi(n)}\sum_{k=a_n+1}^{a_n+\phi(n)}g_n(\xi_{k-1},\xi_k)=\sum_i h(i)\pi(i) \quad a.e. \tag{63}$$

where $\pi^l=(\pi^l(1),\pi^l(2),\cdots)$ *is the unique distribution determined by* P.

Proof. Since periodic strong ergodicity implies $(a_n,\phi(n))$-strong ergodicity. Let $d=1$ in Theorem 1, it follows. □

Now we consider, based on theorem 1,the generalized Shannon-McMillan theorem, we give the following theorem:

Theorem 2. *Let* $(\eta_n)_{n=0}^{\infty}$ *be an* $(a_n,\phi(n))$*-asymptotic circular Markov chain on the state* $\mathcal{X}=\{1,2,\cdots,b\}$ *with the following the initial distribution and the transition matrices resp.,*

$$f^{(0)}=(p(1),p(2),\cdots,p(b)), \tag{64}$$

$$P_n=(p_n(i,j))_{b\times b}, \quad n\geq 1. \tag{65}$$

and $q^l(i,j)$ *be the* (i,j) *element of* $Q^l(l=1,2,\cdots,d)$*. Denote*

$$h_n(i)=-\sum_{j\in\chi}p_n(i,j)\log p_n(i,j) \tag{66}$$

$$h^l(i)=-\sum_{j\in\chi}q_l(i,j)\log q_l(i,j) \tag{67}$$

Let h_n *be column vector with ith element* $h_n(i)$*,* $h_l(l=1,2,\cdots,d)$ *be* d *column vectors with ith elements* $h_l(i)$*. Let* $R_l(l=1,2,\cdots,d)$ *be the same as in Lemma 1 and* R_1 *be* $(a_n,\phi(n))$*-strongly ergodic. If*

$$\lim_{n\to\infty}\frac{1}{\phi(n)}\sum_{t=a_n+1}^{a_n+\phi(n)}\|h_{td+l}-h^l\|=0, \quad l=1,2,\cdots,d, \tag{68}$$

if $\|h_n\|$ *and* $\|h^l\|$ *is finite. Then*

$$\lim_n f_{a_n,\phi(n)}(\omega)=-\sum_{i=1}^{b}\frac{1}{d}\sum_{l=1}^{d}\pi^l(i)\sum_{j=1}^{b}q_l(i,j)\log(q_l(i,j)) \tag{69}$$

where $\pi^l=(\pi^l(1),\pi^l(2),\cdots,\pi^l(b))$ *is the the unique stationary distribution determined by* $R_l(l=1,2,\cdots,d)$*.*

Proof. Let $g_n(x,y)=-\log p_n(x,y)$ in Theorem 1, By (11) and the Lemma 2 of[10], we have for $l=1,2,\cdots,d$

$$\lim_{n\to\infty}\frac{1}{\phi(n)}\sum_{t=a_n+1}^{a_n+\phi(n)}|p_{td+l}(i,j)\log p_{td+l}(i,j)-q_l(i,j)\log q_l(i,j)|=0, \quad \forall i,j\in\chi. \tag{70}$$

By (70), (68) holds.

$$\lim_{n\to\infty}\frac{1}{\phi(n)}\sum_{k=a_n+1}^{a_n+\phi(n)}|p_{kd+l}(i,j)-q_l(i,j)|=0 \tag{71}$$

(71) is equivalent to (11). Since

$$\max_{0\le x\le 1}\{x(\log x)^2\}=4e^{-2} \tag{72}$$

thus

$$E[\log p_n(\xi_{k-1},\xi_k)]^2=\sum_{i=1}^{b}\sum_{j=1}^{b}p_n(i,j)[\log p_n(i,j)]^2P(\xi_{n-1}=i)\le 4be^{-2} \tag{73}$$

By (73), (41) and Theorem 1, (69) follows. □

Theorem 3. *Let$\{\xi_n\}_{n=1}^{\infty}$ be an $(a_n,\phi(n))$-asymptotic circular Markov chain ,and let R_l be defined as in Lemma 7, R_l be $(a_n,\phi(n))$-strong ergodicity,$(\pi_1^l,\pi_2^l,\cdots,\pi_b^l)$ is the unique stationary distribution determined by the stochastic matrix R_l. Let $\{\eta_n,n\ge 1\}$ be an asymptotic circular Markov chain with initial distribution* (15) *and transition matrices*(16) *under measure Q, If $H_l=(h_l(i,j)),l=1,2,\cdots,d,i,j\in\mathcal{X}$ are strictly positive transition matrices,then*

$$\mathcal{L}(\omega)=\sum_{i=1}^{b}\sum_{l=1}^{d}\sum_{j=1}^{b}\frac{\pi^l(i)}{d}t_l(i,j)\log\frac{t_l(i,j)}{h_l(i,j)} \tag{74}$$

Proof. By (1.20)and (2.21),we have

$$\begin{aligned}
&\left|\mathcal{L}_{a_n,\phi(n)}(\omega)-\sum_{l=1}^{d}\sum_{i=1}^{b}\sum_{j=1}^{b}\frac{\pi_i^l}{d}t_l(i,j)\log\frac{t_l(i,j)}{h_l(i,j)}\right|\\
&=\left|\frac{1}{\phi(n)}\sum_{k=a_n+1}^{a_n+\phi(n)}\sum_{j=1}^{b}p_k(\xi_{k-1},j)\log\frac{p_k(\xi_{k-1},j)}{q_k(\xi_{k-1},j)}-\sum_{l=1}^{d}\sum_{i=1}^{b}\sum_{j=1}^{b}\frac{\pi_i^l}{d}t_l(i,j)\log\frac{t_l(i,j)}{h_l(i,j)}\right|\\
&=\left|\frac{1}{\phi(n)}\sum_{k=a_n+1}^{a_n+d[\frac{\phi(n)}{d}]}\sum_{j=1}^{b}p_k(\xi_{k-1},j)\log\frac{p_k(\xi_{k-1},j)}{q_k(\xi_{k-1},j)}+\frac{1}{\phi(n)}\sum_{k=a_n+d[\frac{\phi(n)}{d}]+1}^{a_n+\phi(n)}\sum_{j=1}^{b}p_k(\xi_{k-1},j)\log\frac{p_k(\xi_{k-1},j)}{q_k(\xi_{k-1},j)}\right.\\
&\left.-\sum_{l=1}^{d}\sum_{i=1}^{b}\sum_{j=1}^{b}\frac{\pi_i^l}{d}t_l(i,j)\log\frac{t_l(i,j)}{h_l(i,j)}\right|\\
&\le\left|\frac{1}{\phi(n)}\sum_{j=1}^{b}\sum_{l=1}^{d}\sum_{t=a_n}^{a_n+[\frac{\phi(n)}{d}]-1}p_{td+l}(\xi_{td+l-1},j)\log\frac{p_k(\xi_{td+l-1},j)}{q_k(\xi_{td+l-1},j)}-\sum_{l=1}^{d}\sum_{i=1}^{b}\sum_{j=1}^{b}\frac{\pi_i^l}{d}t_l(i,j)\log\frac{t_l(i,j)}{h_l(i,j)}\right|\\
&+\left|\frac{1}{\phi(n)}\sum_{k=a_n+1}^{a_n+d[\frac{\phi(n)}{d}]}\sum_{j=1}^{b}p_k(\xi_{k-1},j)\log\frac{p_k(\xi_{k-1},j)}{q_k(\xi_{k-1},j)}\right|\\
&\le\left|\frac{1}{\phi(n)}\sum_{j=1}^{b}\sum_{l=1}^{d}\sum_{t=a_n}^{a_n+[\frac{\phi(n)}{d}]-1}\sum_{i=1}^{b}\mathbf{1}_i(\xi_{td+l-1})p_{td+l}(i,j)\log\frac{p_{td+l}(i,j)}{q_{td+l}(i,j)}\right.\\
&\left.-\frac{1}{\phi(n)}\sum_{j=1}^{b}\sum_{l=1}^{d}\sum_{t=a_n}^{a_n+[\frac{\phi(n)}{d}]-1}\sum_{i=1}^{b}\mathbf{1}_i(\xi_{td+l-1})t_l(i,j)\log\frac{t_l(i,j)}{h_l(i,j)}\right|\\
&+\left|\frac{1}{\phi(n)}\sum_{j=1}^{b}\sum_{l=1}^{d}\sum_{t=a_n}^{a_n+[\frac{\phi(n)}{d}]-1}\sum_{i=1}^{b}\mathbf{1}_i(\xi_{td+l-1})t_l(i,j)\log\frac{t_l(i,j)}{h_l(i,j)}-\sum_{l=1}^{d}\sum_{i=1}^{b}\sum_{j=1}^{b}\frac{\pi_i^l}{d}t_l(i,j)\log\frac{t_l(i,j)}{h_l(i,j)}\right|\\
&+\left|\frac{1}{\phi(n)}\sum_{k=a_n+1}^{a_n+d[\frac{\phi(n)}{d}]}\sum_{j=1}^{b}p_k(\xi_{k-1},j)\log\frac{p_k(\xi_{k-1},j)}{q_k(\xi_{k-1},j)}\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\phi(n)}\sum_{j=1}^{b}\sum_{l=1}^{d}\sum_{t=a_n}^{a_n+[\frac{\phi(n)}{d}]-1}\sum_{i=1}^{b}\mathbf{1}_i(\xi_{td+l-1})\left|p_{td+l}(i,j)\log\frac{p_{td+l}(i,j)}{q_{td+l}(i,j)}-t_l(i,j)\log\frac{t_l(i,j)}{h_l(i,j)}\right|\\
&+\sum_{l=1}^{d}\sum_{i=1}^{b}\sum_{j=1}^{b}\left|\frac{1}{\phi(n)}\sum_{t=a_n}^{a_n+\frac{\phi(n)}{d}-1}\mathbf{1}_i(\xi_{td+l-1})-\frac{\pi_i^l}{d}\right|\cdot\left|t_l(i,j)\log\frac{t_l(i,j)}{h_l(i,j)})\right|\\
&+\left|\frac{1}{\phi(n)}\sum_{j=1}^{b}\sum_{k=a_n+d[\frac{\phi(n)}{d}]+1}^{a_n+\phi(n)}p_k(\xi_{k-1},j)\log\frac{p_k(\xi_{k-1},j)}{q_k(\xi_{k-1},j)}\right|\\
&\leq \sum_{j=1}^{b}\sum_{i=1}^{b}\sum_{l=1}^{d}\frac{[\frac{\phi(n)}{d}]-1}{\phi(n)}\cdot\frac{1}{[\frac{\phi(n)}{d}]-1}\sum_{t=a_n}^{a_n+[\frac{\phi(n)}{d}]-1}\left|p_{td+l}(i,j)\log\frac{p_{td+l}(i,j)}{q_{td+l}(i,j)}-t_l(i,j)\log\frac{t_l(i,j)}{h_l(i,j)}\right|\\
&+\sum_{l=1}^{d}\sum_{i=1}^{b}\sum_{j=1}^{b}\left|\frac{1}{\phi(n)}\sum_{t=a_n}^{a_n+\frac{\phi(n)}{d}-1}\mathbf{1}_i(\xi_{td+l-1})-\frac{\pi_i^l}{d}\right|\cdot\left|t_l(i,j)\log\frac{t_l(i,j)}{h_l(i,j)})\right| \qquad (75)\\
&+\left|\frac{1}{\phi(n)}\sum_{j=1}^{b}\sum_{k=a_n+d[\frac{\phi(n)}{d}]+1}^{a_n+\phi(n)}p_k(\xi_{k-1},j)\log\frac{p_k(\xi_{k-1},j)}{q_k(\xi_{k-1},j)}\right|\\
&\leq \sum_{j=1}^{b}\sum_{i=1}^{b}\sum_{l=1}^{d}\frac{[\frac{\phi(n)}{d}]-1}{\phi(n)}\cdot\frac{1}{[\frac{\phi(n)}{d}]-1}\sum_{t=a_n}^{a_n+[\frac{\phi(n)}{d}]-1}\left|p_{td+l}(i,j)\log\frac{p_{td+l}(i,j)}{q_{td+l}(i,j)}-t_l(i,j)\log\frac{t_l(i,j)}{h_l(i,j)}\right|\\
&+\sum_{l=1}^{d}\sum_{i=1}^{b}\sum_{j=1}^{b}\left|\frac{S^l_{a_n,\phi(n)}(i,\omega)}{\phi(n)}-\frac{\pi_i^l}{d}\right|\cdot\left|t_l(i,j)\log\frac{t_l(i,j)}{h_l(i,j)})\right|\\
&+\left|\frac{1}{\phi(n)}\sum_{j=1}^{b}\sum_{k=a_n+d[\frac{\phi(n)}{d}]+1}^{a_n+\phi(n)}p_k(\xi_{k-1},j)\log\frac{p_k(\xi_{k-1},j)}{q_k(\xi_{k-1},j)}\right|
\end{aligned}
$$

By Defnition 2, it is easy to see that $\{(p_k(i,j),q_k(i,j))\}$ absolute mean converge to $(t_l(i,j),h_l(i,j))$. Letting $\varphi(x,y)=x\log\frac{x}{y}$ (suppose $(\varphi(0,y)=0)$ in Lemma 5, we can easily prove $\varphi(x,y)$ is continuous at $(t_l(i,j),h_l(i,j))$, we have by (1.11) and (1.12)

$$
\lim_{n\to\infty}\frac{1}{\phi(n)}\sum_{t=a_n}^{a_n+\phi(n)}\left|p_{td+l}(i,j)\log\frac{p_{td+l}(i,j)}{q_{td+l}(i,j)}-t_l(i,j)\log\frac{t_l(i,j)}{h_l(i,j)}\right|=0. \qquad (76)
$$

Since

$$
\begin{aligned}
\left|p_k(\xi_{k-1},j)\log\frac{p_k(\xi_{k-1},j)}{q_k(\xi_{k-1},j)}\right| &= |p_k(\xi_{k-1},j)\log p_k(\xi_{k-1},j)-p_k(\xi_{k-1},j)\log q_k(\xi_{k-1},j)|\\
&\leq \frac{1}{e}-\log\tau.
\end{aligned} \qquad (77)
$$

Hence

$$
\lim_{n\to\infty}\frac{1}{\phi(n)}\sum_{j=1}^{b}\sum_{k=a_n+d[\frac{\phi(n)}{d}]+1}^{a_n+\phi(n)}\left|p_k(\xi_{k-1},j)\log\frac{p_k(\xi_{k-1},j)}{q_k(\xi_{k-1},j)}\right|=0 \qquad (78)
$$

(34) follows from (33), (36) and (37). □

Corollary 4. *Let P and Q be two measure on $(\Omega,\mathcal{F})$. Let $(\xi_n)_{n=1}^{\infty}$ be a nonhomogeneous Markov chain under measure P. Let $\tilde{P}=(p(i,j)),i,j\in\mathcal{X}$ be a transition matrix and let $\tilde{P}$ be irreducible. If*

$$
\lim_{n\to\infty}\frac{1}{\phi(n)}\sum_{k=a_n}^{a_n+\phi(n)}|p_k(i,j)-p(i,k)|=0,\forall i,j\in S \qquad (79)
$$

$(\pi_1, \pi_2, \ldots, \pi_b)$ *is the unique stationary distribution determined by the stochastic matrix* $\tilde{P}$. *Let* $(\xi_n)_{n=0}^{\infty}$ *be a nonhomogeneous Markov chain with initial distribution* (15) *and transition matrices* (16) *under measure* $\tilde{Q}$, *(2.17) holds. Let*

$$\tilde{Q} = (q(i,j)), \quad q(i,j) > 0, \quad i,j \in \mathcal{X}, \tag{80}$$

be another transition matrix, if for any $i,j \in \mathcal{X}, \{q_n(i,j), n \geq 0\}$ *absolute mean converges to* $q(i,j)$, *that is,*

$$\lim_{n\to\infty} \frac{1}{\phi(n)} \sum_{k=a_n}^{a_n+\phi(n)} |q_k(i,j) - q(i,k)| = 0, \forall i,j \in S \tag{81}$$

then

$$\mathcal{L}(\omega) = \sum_{i=1}^{b}\sum_{j=1}^{b} \pi(i)p(i,j)\log\frac{p(i,j)}{q(i,j)}, \quad a.e.. \tag{82}$$

Proof. It is easy to see that irreducible implies $(a_n, \phi(n))$-strong-strong ergodicity. Letting $d = 1$ in Theorem 3, this corollary follows. □

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